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Supersymmetry and potentials with bound states at arbitrary energies: II

C V Sukumar

Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

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Abstract. It has been shown previously that a potential $V_0(x)$ in one dimension which supports no bound states may be used as a reference potential from which, by successive applications of the concept of a supersymmetric partner to a given Hamiltonian, it is possible to find a potential $V_n(x)$ which supports any specified number n of bound states at any chosen energies E_j, $j=1, \ldots, n$. The reflection coefficient of V_n is related to the reflection coefficient of V_0. Various alternative representations of the potentials constructed by this procedure are presented. An illustrative example in which V_n is constructed by using a $\operatorname{sech}^2 x$ barrier as the reference potential is discussed.

1. Introduction

The algebra of supersymmetry may be used to study the spectral properties of two related Hamiltonians paired together into a single system (Witten 1981). The existence of a conserved charge associated with supersymmetry in supersymmetric quantum mechanics implies certain specific relations between the spectral properties of the two members that form the supersymmetric pair. Examples of supersymmetric systems are discussed in a number of recent reports (for example, Bernstein and Brown 1984, Khare and Maharana 1984, Yamagishi 1984, Ui 1984, d'Hoker and Vinet 1984, Andrianov *et al* 1984, Kostelecky and Nieto 1984, Sukumar 1985a, Blockley and Stedman 1985).

The simplest non-trivial realisation of the algebra of supersymmetry has been shown (Andrianov *et al* 1984, Sukumar 1985b) to lead to the result that every one-dimensional Hamiltonian H can have a supersymmetric partner $\tilde{H}$ such that one of the following spectral features is realised: either (i) $\tilde{H}$ has the same spectrum of eigenvalues as H except for missing the ground state of H, (ii) H has the same spectrum of eigenvalues as $\tilde{H}$ except for missing the ground state of $\tilde{H}$, or (iii) the spectra of H and $\tilde{H}$ are identical. Explicit procedures for finding $\tilde{H}$ with any one of the abovementioned spectral features starting from a specific H have been given previously (Sukumar 1985b). It has been shown that the well known mathematical apparatus of the inverse scattering method (Gelfand and Levitan 1955) may be constructed using the supersymmetric transformations as building blocks (Sukumar 1985c). As an application of this idea it is shown in Sukumar (1986, hereafter referred to as I) that, starting from a reference potential V_0 which does not support any bound states, it is possible to construct a potential V_n which supports bound states at n freely chosen energies E_j, $j=1, \ldots, n$. For positive energies the reflection coefficient of V_n is related to the

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reflection coefficient of V_0 by a multiplicative factor. It is shown in I that V_n has vanishing reflection coefficient and a simple structure when the reference potential is reflectionless, i.e. when the reference system is the free-particle Hamiltonian. The reflectionless V_n has been shown to be related to multi-soliton solutions of the Korteweg-de Vries non-linear equation (see also Kwong and Rosner 1985). The main purpose of this paper is to present some of the properties of V_n when the reference potential V_0 has non-vanishing reflection coefficient.

The layout of the paper is as follows: § 2 provides a summary of the results obtained in I for the structure of V_n. The reduction of V_n to an easily calculable form and various possible representations of V_n when V_0 is allowed to have any reflection coefficient are presented in § 3. The normalisation of the eigenstates of a symmetric V_n constructed from a symmetric V_0 is discussed in § 4. The results from § 3 are used in § 5 to construct V_n when V_0 is a $\operatorname{sech}^2 x$ barrier. Section 6 contains the conclusions.

2. Summary of results obtained in I

Let $V_0(x)$, $-\infty < x < +\infty$, be a potential that supports no bound states. Let $R_0(k)$ be the reflection coefficient for positive energies $E = k^2/2\mu$ where μ is the reduced mass. A potential $V_1(x)$ which supports a single bound state at $E_1 = -\gamma_1^2/2\mu$ and has reflection coefficient

$$R_1(k) = R_0(k)(\gamma_1 - \mathrm{i}k)/(\gamma_1 + \mathrm{i}k) \tag{1}$$

may be found by following the procedure discussed in Sukumar (1985b). It is proved in appendix 1 that for energies below the ground state of any potential it is always possible to find a nodeless but non-normalisable solution of the Schrödinger equation. Let $\psi_0(E_1)$ be a nodeless non-normalisable solution of the Schrödinger equation for V_0 at energy E_1. It is then possible to express the Hamiltonian H_0 corresponding to V_0 in terms of $\psi_0(E_1)$ in the form

$$H_0 = A_0^+(E_1)A_0^-(E_1) + E_1 \tag{2a}$$

where

$$A_0^\pm(E_1) = (2\mu)^{-1/2}\{\pm \mathrm{d}/\mathrm{d}x + [(\mathrm{d}/\mathrm{d}x)\ln \psi_0(E_1)]\}. \tag{2b}$$

The supersymmetric partner to H_0 is given by

$$H_1 = A_0^-(E_1)A_0^+(E_1) + E_1 = H_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \psi_0(E_1). \tag{3}$$

H_1 has a ground state at energy E_1 with eigenfunction

$$\psi_1(E_1) = 1/\psi_0(E_1) \tag{4}$$

and positive energy reflection coefficient $R_1(k)$ given by (1). This property of H_1 has been shown to be a consequence of the feature that $(H_0 - E_1)$ and $(H_1 - E_1)$ are the diagonal elements of a supersymmetric Hamiltonian given by the anti-commutator

$$\mathcal{H} = \{Q, Q^\dagger\} \tag{5}$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ A_0^-(E_1) & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & A_0^+(E_1) \\ 0 & 0 \end{pmatrix} \tag{6}$$

and

$$[Q, \mathscr{H}] = 0 = [Q^{\dagger}, \mathscr{H}]. \tag{7}$$

By iterating this procedure it is shown in I that the potential V_n with bound states at energies

$$E_j = -\gamma_j^2/2\mu \qquad j = 1, 2, \ldots, n, \ \gamma_{j+1} > \gamma_j \tag{8}$$

and positive energy reflection coefficient

$$R_n(k) = R_0(k) \prod_j [(\gamma_j - \mathrm{i}k)/(\gamma_j + \mathrm{i}k)] \tag{9}$$

may be represented by

$$V_n = V_0 - \frac{1}{\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \det D \tag{10}$$

$$D_{ij} = \frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}} \psi_0(E_j) \qquad i, j = 1, \ldots, n \tag{11}$$

where $\psi_0(E_j)$ are non-normalisable solutions of

$$\ddot{\psi}_0(E_j = (\gamma_j^2 + 2\mu V_0)\psi_0(E_j) \qquad j = 1, \ldots, n. \tag{12}$$

$\psi_0(E_j)$ may be chosen to be nodeless for odd values of j and to have a single node for even values of j (see appendix 1 for proof of the existence of such solutions). This choice of $\psi_0(E_j)$ ensures that the determinant of D is nodeless. It is also shown in I that the eigenstates of V_n are given by

$$\psi_n(E_j) = B_{jn} \qquad j = 1, \ldots, n \tag{13a}$$

where

$$B = D^{-1}. \tag{13b}$$

The properties of the reflectionless V_n constructed from a reflectionless V_0, namely $V_0 = 0$, are discussed in I. The following sections consider V_n in the general case with non-vanishing reflection coefficient $R_n(k)$.

3. Representations of V_n

The representation of V_n given by (10) and (11) may be further simplified by using the Schrödinger equation for $\psi_0(E_j)$ to write the elements of the matrix D in the form

$$D_{ij} = \frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}} \psi_0(E_j) = \frac{\mathrm{d}^{i-3}}{\mathrm{d}x^{i-3}} (\gamma_j^2 + 2\mu V_0)\psi_0(E_j) \qquad i > 2. \tag{14}$$

Use of the Leibniz rule to expand the multiple derivative of a product leads to

$$D_{ij} = (\gamma_j^2 + 2\mu V_0) D_{i-2j} + 2\mu \dot{V}_0 (i-3) D_{i-3j} + 2\mu \ddot{V}_0 \frac{(i-3)(i-4)}{2} D_{i-4j} + \ldots + 2\mu D_{1j} \frac{\mathrm{d}^{i-3}}{\mathrm{d}x^{i-3}} V_0 \qquad i > 2. \tag{15}$$

The determinant of a matrix M remains invariant when all the elements in a particular row i are subjected to the transformation $M_{ij} \to M_{ij} + \Sigma_{k \neq i} \alpha_k M_{kj}$ with coefficients α_k

which are independent of j. By repeated applications of the rule on the invariance of a determinant under transformations of the type referred to above, it is easy to show, using equations such as (15) for D_{ij}, D_{i-1j}, D_{i-2j}, etc, that

$$\det D = \det \tilde{D} \tag{16}$$

where $\tilde{D}$ has elements given by

$$\tilde{D}_{ij} = \begin{cases} \gamma_j^{i-1}\psi_0(E_j) & i \text{ odd} \\ \gamma_j^{i-2}\dot{\psi}_0(E_j) & i \text{ even} \end{cases} \qquad i, j = 1, 2, \ldots, n. \tag{17}$$

By applying a similar argument it is easy to show that

$$B_{ij} = (D^{-1})_{ij} = (\tilde{D}^{-1})_{ij} \qquad i, j = 1, \ldots, n \tag{18}$$

since the ij element of the inverse of a matrix is expressible as the ratio of the determinant of the cofactor of D_{ji} and the determinant of D. The potential with n bound states can thus be represented as

$$V_n = V_0 - \frac{1}{\mu}\frac{d^2}{dx^2}\ln\det \tilde{D}. \tag{19}$$

The eigenstates at energies E_j are given by

$$\psi_n(E_j) = (\tilde{D}^{-1})_{jn} = B_{jn} \qquad j = 1, \ldots, n. \tag{20}$$

(18) shows that these eigenfunctions satisfy

$$\sum_j D_{ij}\psi_n(E_j) = \sum_j \tilde{D}_{ij}\psi_n(E_j) = \delta_{in} \qquad i = 1, \ldots, n. \tag{21}$$

It is shown in appendix 2 that

$$\frac{d^2}{dx^2}\ln\det D = 2b_n - \sum_j (\gamma_j^2 + 2\mu V_0) - c_n^2 \tag{22}$$

where

$$c_n = \sum_j \dot{D}_{nj}B_{jn} \tag{23}$$

and

$$b_n = \sum_j \ddot{D}_{nj}B_{jn}. \tag{24}$$

To proceed further we define

$$\beta_j = \gamma_j^2 \qquad \alpha_j = \beta_j + 2\mu V_0 \tag{25a}$$

$$f_j = D_{1j}B_{jn} = \psi_0(E_j)\psi_n(E_j) \tag{25b}$$

$$g_j = D_{2j}B_{jn} = \dot{\psi}_0(E_j)\psi_n(E_j) \tag{25c}$$

for $j = 1, \ldots, n$, and consider the odd and even values of n separately.

(a) *n odd*

It is possible to show that (15), (17) and (21) lead to

$$\sum_j \beta_j^k f_j = 0 = \sum_j \alpha_j^k f_j \qquad k = 0, 1, \ldots, \tfrac{1}{2}(n-3) \tag{26a}$$

$$\sum_j \beta_j^k g_j = 0 = \sum_j \alpha_j^k g_j \qquad k = 0, 1, \ldots, \tfrac{1}{2}(n-3) \tag{26b}$$

$$\sum_j \beta_j^{(n-1)/2} f_j = 1 = \sum_j \alpha_j^{(n-1)/2} f_j. \tag{26c}$$

Using (13), (15) and (23)-(25) it is easy to show that

$$c_n=\sum_j \alpha_j^{(n-1)/2}g_j=\sum_j \beta_j^{(n-1)/2}g_j \tag{27}$$

$$b_n=\sum_j \alpha_j^{(n+1)/2}f_j=\sum_j \beta_j^{(n+1)/2}f_j+(n+1)\mu V_0. \tag{28}$$

Using (16) and (26)-(28), (22) may be simplified to the equivalent forms

$$\frac{d^2}{dx^2}\ln\det\tilde{D}=\sum_j \alpha_j^{(n-1)/2}\left(\alpha_j-\sum_{k\neq j}\alpha_k\right)f_j-\left(\sum_j \alpha_j^{(n-1)/2}g_j\right)^2 \tag{29}$$

or

$$\frac{d^2}{dx^2}\ln\det\tilde{D}=2\mu V_0+\sum_j \beta_j^{(n-1)/2}\left(\beta_j-\sum_{k\neq j}\beta_k\right)f_j-\left(\sum_j \beta_j^{(n-1)/2}g_j\right)^2. \tag{30}$$

Appendix 3 shows that when (26) are valid, then

$$\sum_j \alpha_j^{(n-1)/2}f_j\left(\alpha_j-\sum_{k\neq j}\alpha_k\right)=\sum_j \alpha_j f_j^2\left(\prod_{k\neq j}(\alpha_j-\alpha_k)\right) \tag{31a}$$

$$\left(\sum_j \alpha_j^{(n-1)/2}g_j\right)^2=\sum_j g_j^2\left(\prod_{k\neq j}(\alpha_j-\alpha_k)\right). \tag{31b}$$

Equations (31) are also valid when α_j and α_k are replaced by β_j and β_k. Hence (19) and (29)-(31) show that V_n may be represented in the equivalent forms given by

$$V_n=V_0-\frac{1}{\mu}\sum_j\left(\prod_{k\neq j}(\gamma_j^2-\gamma_k^2)\right)[(\gamma_j^2+2\mu V_0)f_j^2-g_j^2] \tag{32}$$

and

$$V_n=-V_0-\frac{1}{\mu}\sum_j\left(\prod_{k\neq j}(\gamma_j^2-\gamma_k^2)\right)(\gamma_j^2 f_j^2-g_j^2). \tag{33}$$

(*b*) *n even*

It is possible to show that (15), (17) and (21) lead to

$$\sum_j \beta_j^k f_j=0=\sum_j \alpha_j^k f_j \qquad k=0,1,\ldots,\tfrac{1}{2}(n-2) \tag{34a}$$

$$\sum_j \beta_j^k g_j=0=\sum_j \alpha_j^k g_j \qquad k=0,1,\ldots,\tfrac{1}{2}(n-4) \tag{34b}$$

$$\sum_j \beta_j^{(n-2)/2}g_j=1=\sum_j \alpha_j^{(n-2)/2}g_j. \tag{34c}$$

Using (13), (15) and (23)-(25) it is easy to show that

$$c_n=\sum_j \alpha_j^{n/2}f_j=\sum_j \beta_j^{n/2}f_j \tag{35}$$

$$b_n=\sum_j \alpha_j^{n/2}g_j=\sum_j \beta_j^{n/2}g_j+n\mu V_0. \tag{36}$$

Using (16) and (34)-(36), (22) may be simplified to the equivalent forms

$$\frac{d^2}{dx^2}\ln\det\tilde{D}=\sum_j \alpha_j^{(n-2)/2}\left(\alpha_j-\sum_{k\neq j}\alpha_k\right)g_j-\left(\sum_j \alpha_j^{n/2}f_j\right)^2 \tag{37}$$

or

$$\frac{d^2}{dx^2}\ln\det\tilde{D}=\sum_j \beta_j^{(n-2)/2}\left(\beta_j-\sum_{k\neq j}\beta_k\right)g_j-\left(\sum_j \beta_j^{n/2}f_j\right)^2. \quad (38)$$

Appendix 3 shows that when (34) are valid, then

$$\sum_j \alpha_j^{(n-2)/2}\left(\alpha_j-\sum_{k\neq j}\alpha_k\right)g_j=\sum_j g_j^2\left(\prod_{k\neq j}(\alpha_j-\alpha_k)\right) \quad (39a)$$

$$\left(\sum_j \alpha_j^{n/2}f_j\right)^2=\sum_j \alpha_j f_j^2\left(\prod_{k\neq j}(\alpha_j-\alpha_k)\right). \quad (39b)$$

Equations (39) remain valid when α_j and α_k are replaced by β_j and β_k. Hence (19) and (37)-(39) show that V_n may be written in the equivalent forms given by

$$V_n=V_0-\frac{1}{\mu}\sum_j\left(\prod_{k\neq j}(\gamma_j^2-\gamma_k^2)\right)(g_j^2-\gamma_j^2f_j^2) \quad (40)$$

and

$$V_n=V_0-\frac{1}{\mu}\sum_j\left(\prod_{k\neq j}(\gamma_j^2-\gamma_k^2)\right)[g_j^2-(\gamma_j^2+2\mu V_0)f_j^2]. \quad (41)$$

The odd and even cases may be combined together to give the result that the potential with n bound states at energies $E_j=-\gamma_j^2/2\mu$, $j=1,\ldots,n$, may be represented as

$$V_n=V_0+\frac{1}{\mu}\sum_j(-1)^j\left(\prod_{k\neq j}|\gamma_j^2-\gamma_k^2|\right)\psi_n^2(E_j)[(\gamma_j^2+2\mu V_0)\psi_0^2(E_j)-\dot{\psi}_0^2(E_j)] \quad (42)$$

or alternatively as

$$V_n=(-1)^nV_0+\frac{1}{\mu}\sum_j(-1)^j\left(\prod_{k\neq j}|\gamma_j^2-\gamma_k^2|\right)\psi_n^2(E_j)[\gamma_j^2\psi_0^2(E_j)-\dot{\psi}_0^2(E_j)] \quad (43)$$

where $\psi_n(E_j)$ are the un-normalised eigenstates of V_n given by (20)

$$\psi_n(E_j)=(\tilde{D}^{-1})_{jn} \qquad j=1,\ldots,n.$$

$\psi_n(E_j)$ may be found by inverting the matrix $\tilde{D}$ with elements (equation (21))

$$\tilde{D}_{ij}=\begin{cases}\gamma_j^{i-1}\psi_0(E_j) & i \text{ odd}\\ \gamma_j^{i-2}\dot{\psi}_0(E_j) & i\text{ even}\end{cases} \qquad i,j=1,2,\ldots,n$$

where $\psi_0(E_j)$ are the unnormalisable solutions of the Schrödinger equation for V_0 at energy E_j chosen such that $\psi_0(E_j)$ is nodeless for odd values of j and has a single node for even values of j. It is clear from (42) and (43) that the eigenfunctions $\psi_n(E_j)$ satisfy

$$\sum_j(-1)^{j-1}\left(\prod_{k\neq j}|\gamma_j^2-\gamma_k^2|\right)\psi_0^2(E_j)\psi_n^2(E_j)=\frac{1+(-1)^{n+1}}{2}=\begin{cases}1 & \text{if } n \text{ is odd}\\ 0 & \text{if } n\text{ is even}\end{cases} \quad (44)$$

The reflection coefficient of V_n for positive energies $E_k=k^2/2\mu$ is given by (9):

$$R_n(k)=R_0(k)\prod_j[(\gamma_j-ik)/(\gamma_j+ik)].$$

When $V_0 = 0$ and $\psi_0(E_j)$ are chosen such that

$$\psi_0(E_j) = \begin{cases} \cosh \gamma_j x & j \text{ odd} \\ \sinh \gamma_j x & j \text{ even} \end{cases} \qquad j = 1, 2, \ldots, n \tag{45}$$

then

$$\gamma_j^2 \psi_0^2(E_j) - \dot\psi_0^2(E_j) = (-1)^{j+1} \gamma_j^2 \qquad j = 1, \ldots, n. \tag{46}$$

The resulting symmetric reflectionless potential with n bound states is given by

$$V_n = -\frac{1}{\mu} \sum_j \left(\prod_{k \neq j} |\gamma_j^2 - \gamma_k^2| \right) \gamma_j^2 \psi_n^2(E_j) \tag{47}$$

in agreement with the result obtained in equations (65) and (79) of I.

4. Normalisation of the eigenstates of symmetric V_n

If $V_0(x)$ is a symmetric function of x then a symmetric potential with n bound states can be constructed by choosing the basis functions defining D_{ij} in (17) such that $\psi_0(E_j)$ is an even function of x for odd values of j and an odd function of x for even values of j Let V_n be a symmetric potential so constructed with n bound states at $E_j = -\gamma_j^2/2\mu$, $j = 1, \ldots, n$, and unnormalised eigenfunctions

$$\psi_n(E_j) = [\tilde{D}(\gamma_1, \gamma_2, \ldots, \gamma_n)]_{jn}^{-1} \qquad j = 1, \ldots, n. \tag{48}$$

Let V_{n-1} be a symmetric potential similarly constructed with $(n-1)$ bound states at $E_j = -\gamma_j^2/2\mu$, $j = 1, \ldots, n-1$, and unnormalised eigenfunctions

$$\psi_{n-1}(E_j) = [\tilde{D}(\gamma_1, \gamma_2, \ldots, \gamma_{n-1})]_{jn-1}^{-1} \qquad j = 1, \ldots, n-1. \tag{49}$$

Since V_{ns} and V_{n-1s} are connected by supersymmetry the eigenstates at a common eigenenergy E_j are related by

$$\psi_n(E_j) = \lambda_j \left[\frac{\mathrm{d}}{\mathrm{d}x} + \left(\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_n(E_n) \right) \right] \psi_{n-1}(E_j) \qquad j \neq n \tag{50}$$

as shown by equations (51), (53) and (54) of I. The constants λ_j may be determined as follows. Let

$$\lim_{x \to \infty} \psi_0(E_j) = \varepsilon_j \exp(\gamma_j x) \qquad j = 1, \ldots, n. \tag{51}$$

It is easy to show by evaluating the relevant determinants in the limit $x \to \infty$ that

$$\lim_{x \to \infty} \psi_n(E_j) = \frac{1}{\varepsilon_j} \exp(-\gamma_j x) \Bigg/ \left(\prod_{\substack{k \neq j \\ k = 1, 2, \ldots, n}} (\gamma_k - \gamma_j) \right)^{-1} \qquad j = 1, \ldots, n. \tag{52}$$

Similarly

$$\lim_{x \to \infty} \psi_{n-1}(E_j) = \frac{1}{\varepsilon_j} \exp(-\gamma_j x) \Bigg/ \left(\prod_{\substack{k \neq j \\ k = 1, 2, \ldots, n-1}} (\gamma_k - \gamma_j) \right)^{-1} \qquad j = 1, \ldots, n-1. \tag{53}$$

Consideration of the $x \to \infty$ limit of (50) and use of (52) and (53) gives

$$\lambda_j = (\gamma_n^2 - \gamma_j^2)^{-1} \qquad j = 1, \ldots, n-1. \tag{54}$$

It was shown in equations (50)-(52) of I that the Hamiltonians with n and $(n-1)$ bound states are given by

$$H_n = A^-_{n-1}(E_n)A^+_{n-1}(E_n) + E_n \qquad H_{n-1} = A^+_{n-1}(E_n)A^-_{n-1}(E_n) + E_n \tag{55a}$$

and

$$A^{\pm}_{n-1}(E_n) = (2\mu)^{-1/2}\left[\pm\frac{\mathrm{d}}{\mathrm{d}x} - \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_n(E_n)\right)\right]. \tag{55b}$$

Use of (50), (54) and (55) gives

$$\int_{-\infty}^{\infty}\psi_n^2(E_j)\,\mathrm{d}x = \frac{2\mu}{(\gamma_n^2-\gamma_j^2)^2}\int_{-\infty}^{\infty}[A^-_{n-1}\psi_{n-1}(E_j)]^2\,\mathrm{d}x \qquad j\neq n. \tag{56}$$

But for any bound-state eigenfunctions φ_1 and φ_2

$$\int_{-\infty}^{\infty}\varphi_1 A^-_{n-1}\varphi_2\,\mathrm{d}x = \int_{-\infty}^{\infty}\varphi_2 A^+_{n-1}\varphi_1\,\mathrm{d}x. \tag{57}$$

Using this relation (56) may be written as

$$\int_{-\infty}^{\infty}\psi_n^2(E_j)\,\mathrm{d}x = \frac{2\mu}{(\gamma_n^2-\gamma_j^2)^2}\int_{-\infty}^{\infty}\psi_{n-1}(E_j)A^+_{n-1}A^-_{n-1}\psi_{n-1}(E_j)\,\mathrm{d}x \qquad j\neq n. \tag{58}$$

(55) may now be used to obtain

$$\int_{-\infty}^{\infty}\psi_n^2(E_j)\,\mathrm{d}x = \frac{1}{(\gamma_n^2-\gamma_j^2)}\int_{-\infty}^{\infty}\psi_{n-1}^2(E_j)\,\mathrm{d}x \qquad j\neq n. \tag{59}$$

Iteration of (59) for $j=1$ shows that

$$\int_{-\infty}^{\infty}\psi_n^2(E_1)\,\mathrm{d}x = \left(\prod_{k\neq 1}(\gamma_k^2-\gamma_1^2)\right)^{-1}\int_{-\infty}^{\infty}\psi_1^2(E_1)\,\mathrm{d}x. \tag{60}$$

In (4) we have for $n=1$

$$\psi_1(E_1) = 1/\psi_0(E_1).$$

Hence

$$\int_{-\infty}^{\infty}\psi_n^2(E_1)\,\mathrm{d}x = \left(\prod_{k\neq 1}(\gamma_k^2-\gamma_1^2)\right)^{-1}\int_{-\infty}^{\infty}\mathrm{d}x/\psi_0^2(E_1). \tag{61}$$

When V_0 is a symmetric function of x, $\psi_0(E_j)$ for all odd values of j may be chosen to be the same even function of x expressed as a power series with E_j as a parameter. Examination of the determinants involved in evaluating $(\tilde{D}^{-1})_{jn}$ then shows that $\psi_n(E_{j+2})$ is obtained from $\psi_n(E_j)$ by the substitution $E_j \leftrightarrow E_{j+2}$ in $\tilde{D}$ while all the other energies are left unaltered. By using this symmetry property, the normalisation integral of $\psi_n(E_j)$ for all odd values of j may be evaluated to be

$$\int_{-\infty}^{\infty}\psi_n^2(E_j)\,\mathrm{d}x = \left(\prod_{k\neq j}|\gamma_k^2-\gamma_j^2|\right)^{-1}\int_{\infty}^{\infty}\mathrm{d}x/\psi_0^2(E_j) \qquad j\text{ odd}. \tag{62}$$

The normalisation integral for the states with even values of j may be determined as follows. Iteration of (59) for $j=2$ gives

$$\int_{-\infty}^{\infty}\psi_n^2(E_2)\,\mathrm{d}x = \left(\prod_{k\neq 1,2}(\gamma_k^2-\gamma_2^2)\right)^{-1}\int_{-\infty}^{\infty}\psi_2^2(E_2)\,\mathrm{d}x. \tag{63}$$

It is easy to show from (11) and (13) that for $n=2$

$$\psi_2(E_1) = -\psi_0(E_2)/D_2 \qquad \psi_2(E_2) = \psi_0(E_1)/D_2 \tag{64a}$$

$$D_2 = \psi_0(E_1)\dot\psi_0(E_2) - \dot\psi_0(E_1)\psi_0(E_2). \tag{64b}$$

For a symmetric V_2, $\psi_0(E_1)$ and $\dot\psi_0(E_2)$ are even functions of x while $\psi_0(E_2)$ and $\dot\psi_0(E_1)$ are odd functions of x and have a single node at $x=0$. Noting that $\psi_2(E_2)$ may also be written in the form

$$\psi_2(E_2) = \frac{1}{\dot\psi_0(E_2)}\left(1 + \frac{\psi_0(E_2)\dot\psi_0(E_1)}{D_2}\right) \tag{65}$$

the normalisation integral for $\psi_2(E_2)$ may be given as

$$\int_{-\infty}^{\infty} \psi_2^2(E_2)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\psi_0(E_1)}{\dot\psi_0(E_2)}\frac{1}{D_2}\left(1 + \frac{\psi_0(E_2)\dot\psi_0(E_1)}{D_2}\right). \tag{66}$$

Using

$$D_2 = (\gamma_2^2 - \gamma_1^2)\psi_0(E_1)\psi_0(E_2) \tag{67}$$

it is possible to evaluate the integral in (66) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_2^2(E_2)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \frac{1}{D_2}\left[\frac{\psi_0(E_1)}{\dot\psi_0(E_2)} + \frac{1}{(\gamma_2^2-\gamma_1^2)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\dot\psi_0(E_1)}{\dot\psi_0(E_2)}\right)\right] \\ &= \frac{1}{(\gamma_2^2-\gamma_1^2)}\int_{-\infty}^{\infty} (\gamma_2^2 + 2\mu V_0)/\dot\psi_0^2(E_2)\,\mathrm{d}x.\end{aligned} \tag{68}$$

(63) then gives

$$\int_{-\infty}^{\infty} \psi_n^2(E_2)\,\mathrm{d}x = \left(\prod_{k\neq 2} |\gamma_k^2 - \gamma_2^2|\right)^{-1} \int_{-\infty}^{\infty} \frac{(\gamma_2^2 + 2\mu V_0)}{\dot\psi_0^2(E_2)}\,\mathrm{d}x. \tag{69}$$

By choosing $\psi_0(E_j)$ for all even values of j to be the same odd function of x with E_j as a parameter it is possible to use a symmetry argument as for the case of odd values of j to obtain the normalisation integral of $\psi_n(E_j)$ for all even values of j in the form

$$\int_{-\infty}^{\infty} \psi_n^2(E_j)\,\mathrm{d}x = \left(\prod_{k\neq j} |\gamma_k^2 - \gamma_j^2|\right)^{-1} \int_{-\infty}^{\infty} \frac{(\gamma_j^2 + 2\mu V_0)}{\dot\psi_0^2(E_j)}\,\mathrm{d}x \qquad j \text{ even}. \tag{70}$$

The normalisation integrals for the eigenstates of the symmetric potential V_n are thus given by

$$\int_{-\infty}^{\infty} \psi_n^2(E_j)\,\mathrm{d}x = \left(\prod_{k\neq j} |\gamma_k^2 - \gamma_j^2|\right)^{-1} N_j \qquad j = 1, 2, \ldots, n \tag{71}$$

where

$$N_j = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x/\psi_0^2(E_j) & j \text{ odd} \\ \displaystyle\int_{-\infty}^{\infty} \frac{(\gamma_j^2 + 2\mu V_0)}{\dot\psi_0^2(E_j)}\,\mathrm{d}x & j \text{ even.} \end{cases} \tag{72}$$

For the special case of $V_0=0$ use of (45) for $\psi_0(E_j)$ shows that the eigenfunctions of the symmetric reflectionless potentials obtained from (20) satisfy

$$\int_{-\infty}^{\infty} \psi_n^2(E_j)\,\mathrm{d}x = \frac{2}{\gamma_j}\left(\prod_{k\neq j} |\gamma_k^2-\gamma_j^2|\right)^{-1} \qquad j=1,2,\ldots,n \tag{73}$$

in agreement with the result obtained in equation (A3.24) of I.

5. The $\mathrm{sech}^2 x$ barrier as reference potential

In this section the procedure for the construction of V_n described in § 3 is illustrated by considering the barrier

$$V_0=\sigma\,\mathrm{sech}^2 x \tag{74}$$

as the reference potential. The reflection coefficient $R_0(k)$ of the barrier (74) is given by (Landau and Lifschitz 1965)

$$R_0(k,\sigma)=\frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)}\frac{\Gamma(-\mathrm{i}k-s)}{\Gamma(-s)}\frac{\Gamma(-\mathrm{i}k+s+1)}{\Gamma(s+1)} \tag{75}$$

where

$$s=\tfrac{1}{2}[-1+(1-8\mu\sigma)^{1/2}]. \tag{76}$$

The Schrödinger equation for the barrier (74) for any negative energy $E=-\gamma^2/2\mu$ may be transformed into the differential equation for the associated Legendre functions (Landau and Lifschitz 1965, Gradshteyn and Ryzhik 1965). The solution which is an even function of x is given by

$$\varphi_0(\gamma,x)=\mathrm{sech}^\gamma x\sum_{j=0}^{\infty} a_j\tanh^{2j} x \tag{77a}$$

$$a_0=1 \qquad a_m=a_{m-1}\frac{(\gamma+2m-2)(\gamma+2m-1)+2\mu\sigma}{2m(2m-1)}. \tag{77b}$$

The solution which is an odd function of x is given by

$$\tilde{\varphi}_0(\gamma,x)=\mathrm{sech}^\gamma x\tanh x\sum_{j=0}^{\infty} b_j\tanh^{2j} x \tag{78a}$$

$$b_0=1 \qquad b_m=b_{m-1}\frac{(\gamma+2m-1)(\gamma+2m)+2\mu\sigma}{2m(2m+1)}. \tag{78b}$$

A set of energies $E_j=-\gamma_j^2/2\mu$ may be chosen and the matrix $\tilde{D}$ constructed using (17),

$$\tilde{D}_{ij}=\begin{cases}\gamma_j^{i-1}\psi_0(E_j) & i\text{ odd}\\ \gamma_j^{i-2}\dot{\psi}_0(E_j) & i\text{ even}\end{cases} \qquad i,j=1,2,\ldots,n$$

where

$$\psi_0(E_j)=\begin{cases}\varphi_0(\gamma_j,x) & j\text{ odd}\\ \tilde{\varphi}_0(\gamma_j,x) & j\text{ even}\end{cases} \qquad j=1,2,\ldots,n. \tag{79}$$

The matrix $\tilde{D}$ may then be inverted. The unnormalised eigenstates $\psi_n(E_j)$ of V_n are given by the jn elements of $\tilde{D}^{-1}$. Using these eigenstates the potential V_n may easily be constructed from (42) or (43). The reflection coefficient of V_n is given by (9) with $R_0(k)$ determined by (75) and (76).

Figures 1-4 show potentials V_1 which support a single bound state at a fixed value of the energy, E_1, for various values of the barrier height σ of the reference potential. For $\sigma > 0.125$ a double hump appears in V_1 to accommodate the bound state. The

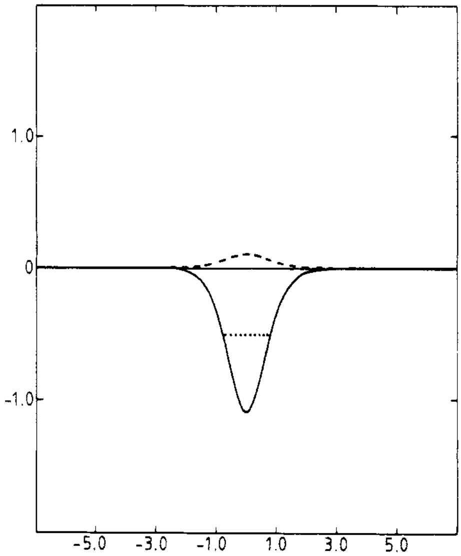

Figure 1. The figure shows a potential with a single bound state at energy $E_1 = -0.5$ constructed from $V_0 = \sigma \operatorname{sech}^2 x$, $\sigma = 0.1$, for $\mu = 1.0$. V_0 is shown by the broken curve. The position of the bound state is indicated by a horizontal dotted line.

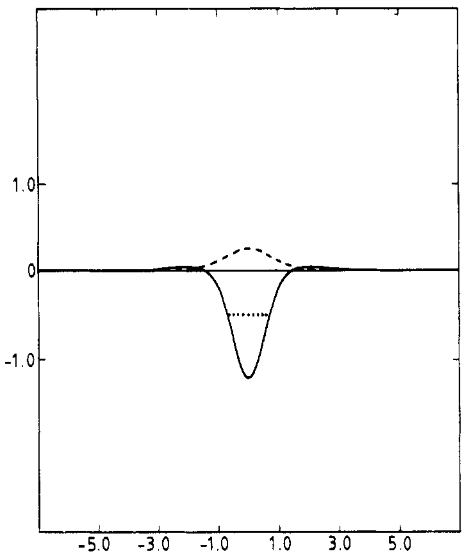

Figure 2. Same as figure 1 but with $\sigma = 0.25$.

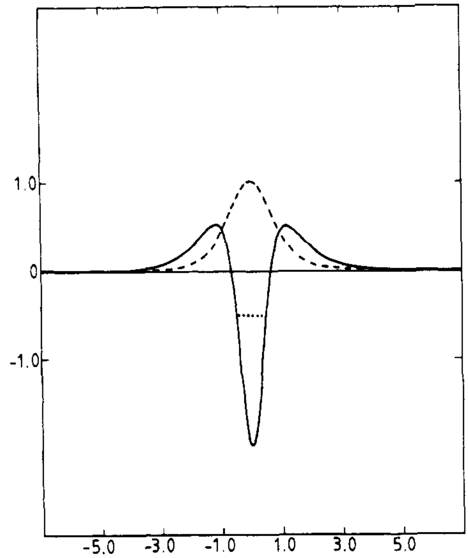

Figure 3. Same as figure 1 but with $\sigma = 1.0$.

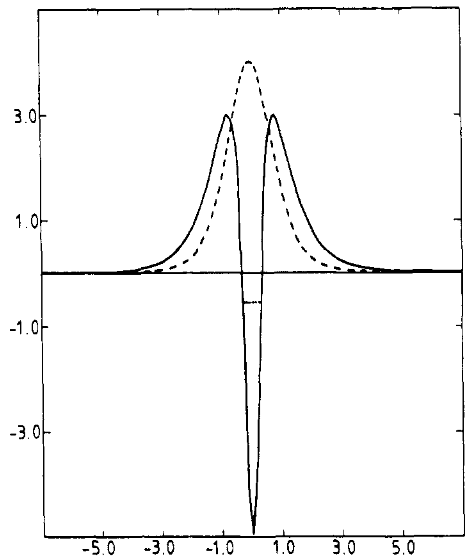

Figure 4. Same as figure 1 but with $\sigma = 4.0$.

hump becomes sharper and the well becomes deeper and narrower as σ increases to higher values.

Figures 5-8 show potentials V_2 which support two bound states at fixed energies E_1 and E_2 for various values of the barrier height of the reference potential. For the values of E_1 and E_2 chosen in figures 5-8 a symmetric double well structure for V_2 is necessary to accomodate the two bound states. For $\sigma > 0.125$ a double hump appears in the wings in addition to the double well and for $\sigma > 1.0$ the walls separating the two wells become steeper and the double wells become narrower and deeper.

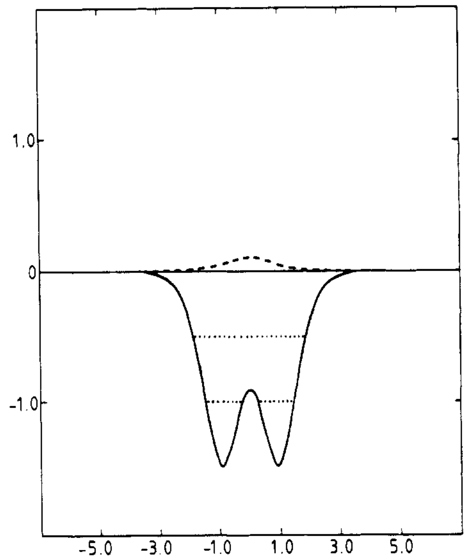

Figure 5. The figure shows a potential with two bound states at energies $E_1 = -0.5$, $E_2 = -1.0$ constructed from $V_0 = \sigma \operatorname{sech}^2 x$, $\sigma = 0.1$, for $\mu = 1.0$. V_0 is shown by the broken curve. The positions of the bound states are indicated by horizontal dotted lines.

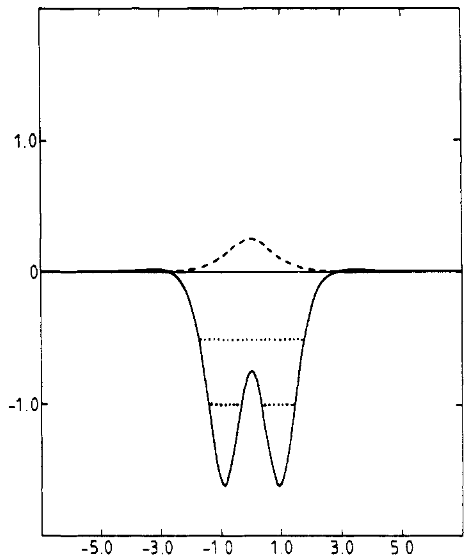

Figure 6. Same as figure 5 but with $\sigma = 0.25$.

6. Conclusions

In this paper it has been shown that the potentials constructed using the algebra of supersymmetry in a step by step fashion by the addition of a single bound state at a time have simple mathematical representations. The scheme outlined in this paper gives a general procedure for the systematic construction of potentials in one dimension

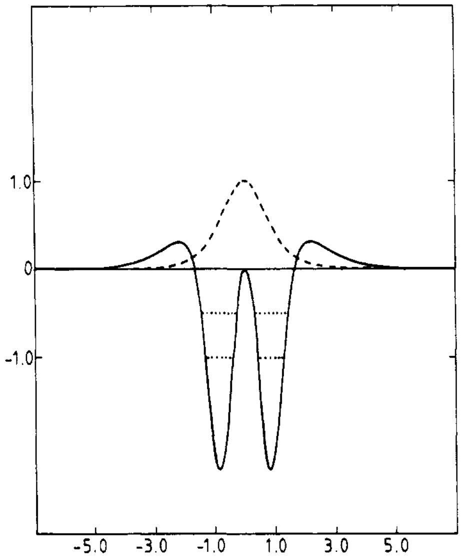

Figure 7. Same as figure 5 but with $\sigma = 1.0$.

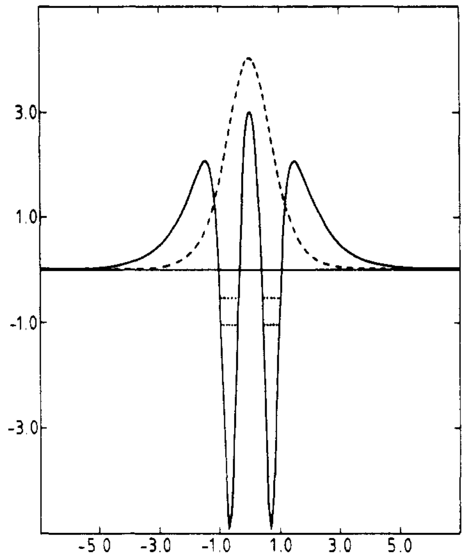

Figure 8. Same as figure 5 but with $\sigma = 4.0$.

with specific spectral properties starting from a reference potential of known reflection coefficient.

Appendix 1

In this appendix it is proved that for energies below the ground state of a potential, the solution of the Schrödinger equation is either nodeless or has a single node.

Let $\varphi_0(x, E_0)$, $E_0 = -\gamma_0^2/2\mu$ be the ground state of a potential V. $\varphi_0(x)$ may be normalised such that $\varphi_0(0)$ is positive. $\varphi_0(x)$ must be a nodeless function. Hence $\varphi_0(x) > 0$ for all finite x. Let

$$\varphi_0(x)|_{x=0} = |a| \qquad \dot{\varphi}_0(x)|_{x=0} = b. \tag{A1.1}$$

Let $\varphi(x)$ be a solution of the Schrödinger equation for an energy $E = -\gamma^2/2\mu$ which lies below the ground state of V, i.e. $\gamma > \gamma_0$. $\varphi(x)$ may be chosen such that

$$\varphi(x)|_{x=0} = |a| \qquad \dot{\varphi}(x)|_{x=0} = b \tag{A1.2}$$

as it is possible to choose these two initial values freely for a Schrödinger-type second-order differential equation. The Schrödinger equations for φ_0 and φ given by

$$\ddot{\varphi}_0 = (\gamma_0^2 + 2\mu V_0)\varphi_0 \tag{A1.3}$$

$$\ddot{\varphi} = (\gamma^2 + 2\mu V_0)\varphi \tag{A1.4}$$

show that

$$(\mathrm{d}/\mathrm{d}x)(\varphi_0\dot{\varphi} - \dot{\varphi}_0\varphi) = (\gamma^2 - \gamma_0^2)\varphi_0\varphi. \tag{A1.5}$$

Integration of (A1.5) using (A1.1) and (A1.2) gives

$$\varphi_0(x)\dot{\varphi}(x) - \dot{\varphi}_0(x)\varphi(x) = (\gamma^2 - \gamma_0^2)\int_0^x \varphi_0(y)\varphi(y)\,\mathrm{d}y \tag{A1.6}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\varphi(x)}{\varphi_0(x)} = (\gamma^2 - \gamma_0^2)\frac{1}{\varphi_0^2(x)}\int_0^x \varphi_0(y)\varphi(y)\,\mathrm{d}y. \tag{A1.7}$$

Integration of (A1.7) using (A1.1) and (A1.2) gives

$$\frac{\varphi(x)}{\varphi_0(x)} - 1 = (\gamma^2 - \gamma_0^2)\int_0^x \frac{\mathrm{d}z}{\varphi_0^2(z)}\int_0^z \varphi_0(y)\varphi(y)\,\mathrm{d}y. \tag{A1.8}$$

Let

$$f(x) = \varphi(x)/\varphi_0(x). \tag{A1.9}$$

Then

$$f(x) = 1 + (\gamma^2 - \gamma_0^2)\int_0^x \frac{\mathrm{d}z}{\varphi_0^2(z)}\int_0^z \varphi_0^2(y)f(y)\,\mathrm{d}y. \tag{A1.10}$$

$\varphi_0(x)$ is positive semi-definite for all x and $\gamma^2 > \gamma_0^2$. These conditions guarantee that $f(x) - 1$ is positive semi-definite for all positive x as iteration of the integral equation (A1.10) shows that

$$f(x) = 1 + (\gamma^2 - \gamma_0^2)\int_0^x \frac{\mathrm{d}z}{\varphi_0^2(z)}\int_0^z \varphi_0^2(y)\,\mathrm{d}y + (\gamma^2 - \gamma_0^2)^2$$
$$\times \int_0^x \frac{\mathrm{d}z}{\varphi_0^2(z)}\int_0^z \varphi_0^2(y)\,\mathrm{d}y \int_0^y \frac{\mathrm{d}z'}{\varphi_0^2(z')}\int_0^{z'} \varphi_0^2(y')\,\mathrm{d}y' + \ldots. \tag{A1.11}$$

The positive definiteness of $f(x) - 1$ for negative x can be established by changing variables from x to $x' = -x$ and reducing (A1.10) to the form

$$f(-x') = 1 + (\gamma^2 - \gamma_0^2)\int_0^{x'} \frac{\mathrm{d}z'}{\varphi_0^2(-z')}\int_0^{z'} \varphi_0^2(-y')f(-y')\,\mathrm{d}y'. \tag{A1.12}$$

Hence

$$f(x) \geqslant 1 \qquad |x| \leqslant \infty \tag{A1.13}$$

and so

$$\varphi(x) \geqslant \varphi_0(x). \tag{A1.14}$$

Since $\varphi_0(x) > 0$, $|x| < \infty$, it follows that $\varphi(x) > 0$, $|x| < \infty$, i.e. $\varphi(x)$ must also be nodeless. A second linearly independent solution of (A1.4) is given by

$$\tilde{\varphi}(x) = \varphi(x) \int_{x_0}^{x} \mathrm{d}y / \varphi^2(y). \tag{A1.15}$$

$\tilde{\varphi}(x)$ has a node at $x = x_0$. $\tilde{\varphi}(x)$ has no other nodes since φ is nodeless and positive for all values of x. It is therefore possible to conclude that for energies below the ground state of a potential it is always possible to find a nodeless solution $\varphi(x)$ and a linearly independent solution $\tilde{\varphi}(x)$ which can be chosen to have a single node at any point $x = x_0$.

Appendix 2

In this appendix a simplified expression for the second derivative of the determinant of D in (11) in the main text is obtained.

It was shown in the main text ((11)-(13)) that

$$D_{ij} = \frac{\mathrm{d}^{i-1}}{\mathrm{d}x^{i-1}} \psi_0(E_j) \qquad i, j = 1, \ldots, n$$

$$\ddot{\psi}_0(E_j) = (\gamma_j^2 + 2\mu V_0)\psi_0(E_j) \qquad E_j = -\gamma_j^2/2\mu$$

and

$$B = D^{-1}.$$

Differentiation of (11) gives

$$\dot{D}_{ij} = (1 - \delta_{in}) D_{i+1j} + \delta_{in} \dot{D}_{nj} \tag{A2.1}$$

and so

$$(\dot{D}B)_{ij} = \delta_{i+1j} + \delta_{in} c_j \tag{A2.2}$$

where

$$c_j = \sum_k \dot{D}_{nk} B_{kj} \qquad j = 1, \ldots, n. \tag{A2.3}$$

It was shown in equation (A2.16) of I that

$$c_n = \sum_k \dot{D}_{nk} B_{kn} = \frac{\mathrm{d}}{\mathrm{d}x} \ln \det D. \tag{A2.4}$$

It is easy to show from (A2.1) that

$$\ddot{D}_{ij} = (1 - \delta_{in})(1 - \delta_{i+1n}) D_{i+2j} + \delta_{i+1n} \dot{D}_{nj} + \delta_{in} \ddot{D}_{nj} \tag{A2.5}$$

therefore

$$(\ddot{D}B)_{ij} = \delta_{i+2j} + \delta_{i+1n} c_j + \delta_{in} b_j \tag{A2.6}$$

where

$$b_j = \sum_k \ddot{D}_{nk} B_{kj} \qquad j = 1, \ldots, n. \tag{A2.7}$$

Hence

$$\mathrm{Tr}(\ddot{D}B) = c_{n-1} + b_n \tag{A2.8}$$

where

$$b_n = \sum_k \ddot{D}_{nk} B_{kn}. \tag{A2.9}$$

Alternatively (14) and (15) in the main text may be used to give

$$\ddot{D}_{ij} = (\gamma_j^2 + 2\mu V_0) D_{ij} + (i-1) 2\mu \dot{V}_0 D_{i-1j}$$
$$+ \frac{(i-1)(i-2)}{2} 2\mu \ddot{V}_0 D_{i-2j} + \ldots + 2\mu \frac{\mathrm{d}^{i-1} V_0}{\mathrm{d}x^{i-1}} D_{1j}. \tag{A2.10}$$

Use of (13) then gives

$$(\ddot{D}B)_{ii} = \sum_j (\gamma_j^2 + 2\mu V_0) D_{ij} B_{ji} \tag{A2.11}$$

and so

$$\mathrm{Tr}(\ddot{D}B) = \sum_j (\gamma_j^2 + 2\mu V_0). \tag{A2.12}$$

Comparison of (A2.8) and (A2.12) shows that

$$b_n + c_{n-1} = \sum_j (\gamma_j^2 + 2\mu V_0). \tag{A2.13}$$

It is easy to show from (13) that

$$\dot{B} = -B\dot{D}B. \tag{A2.14}$$

(A2.14) and (A2.2) then lead to

$$\dot{B}_{jn} = -B_{jn-1} - B_{jn} c_n. \tag{A2.15}$$

Use of (A2.4) provides

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \det D = \sum_j \ddot{D}_{nj} B_{jn} + \sum_j \dot{D}_{nj} \dot{B}_{jn}. \tag{A2.16}$$

(A2.9), (A2.15), (A2.3) and (A2.13) may be used to simplify (A2.16) to the form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \det D = 2b_n - \sum_j (\gamma_j^2 + 2\mu V_0) - c_n^2 \tag{A2.17}$$

where b_n and c_n are given by (A2.4) and (A2.9) respectively.

Appendix 3

In this appendix four different functional relations are established. The following notation will be used throughout this appendix:

$$\sum_{j_1} A_{j_1} \sum_{j_2 \neq j_1} B_{j_2} \sum_{j_3 \neq j_2, j_1} C_{j_3} \ldots \equiv \sum_{j_1} A_{j_1} \sum_{j_2, j_3 \ldots}^{\sim j_1} B_{j_2} C_{j_3} \ldots. \tag{A3.1}$$

Case (*i*). Let n be an odd integer. Let f_j be a set of functions which satisfy

$$\sum_{j=1}^{n} \alpha_j^k f_j = 0 \qquad k = 0, 1, \ldots, \tfrac{1}{2}(n-3). \tag{A3.2}$$

$$\sum_{j=1}^{n} \alpha_j^k f_j = 1 \qquad k = \tfrac{1}{2}(n-1). \tag{A3.3}$$

Let

$$F = \sum_{j=1}^{n} \alpha_j^{(n-1)/2} f_j \left(\alpha_j - \sum_{k \neq j} \alpha_k \right). \tag{A3.4}$$

Using (A3.3) F may be written in the form

$$F = \sum_j \sum_k (\alpha_j \alpha_k)^{(n-1)/2} f_j f_k \left(\alpha_j - \sum_{l \neq j} \alpha_l \right). \tag{A3.5}$$

The terms in the double summation can be regrouped to give

$$F = \sum_j f_j^2 \alpha_j \left(\alpha_j^{n-1} - \alpha_j^{n-2} \sum_{l \neq j} \alpha_l \right) - 2 \sum_j \sum_{k>j} (\alpha_j \alpha_k)^{(n-1)/2} \sum_{l \neq j,k} \alpha_l. \tag{A3.6}$$

Let

$$\begin{aligned} F_1 &= \sum_{j=1}^{n} f_j^2 \alpha_j \prod_{k \neq j} (\alpha_j - \alpha_k) \\ &= \sum_j f_j^2 \alpha_j \left(\alpha_j^{n-1} - \alpha_j^{n-2} \sum_{k \neq j} \alpha_k + \alpha_j^{n-3} \sum_{kl}^{\sim j} \alpha_k \alpha_l + \ldots + \sum_{k \neq j} \alpha_k \right). \end{aligned} \tag{A3.7}$$

Then

$$\begin{aligned} F_1 - F = &\left\{ 2 \sum_j \sum_{k>j} (\alpha_j \alpha_k)^{(n-1)/2} f_j f_k \sum_{l \neq j,k} \alpha_l + \sum_j f_j^2 \alpha_j^{n-2} \sum_{kl}^{\sim j} \alpha_k \alpha_l \right\} \\ &- \sum_j f_j^2 \alpha_j \left(\alpha_j^{n-4} \sum_{klm}^{\sim j} \alpha_k \alpha_l \alpha_m - \ldots - \prod_{k \neq j} \alpha_k \right). \end{aligned} \tag{A3.8}$$

To proceed further consider

$$\left(\sum_j \alpha_j^{(n-3)/2} f_j \right) \left(\sum_j \alpha_j^{(n-1)/2} f_j \sum_{kl}^{\sim j} \alpha_k \alpha_l \right) = 0 \tag{A3.9}$$

which is valid since the terms within the first set of brackets vanish as shown by (A3.2). Expanding (A3.9) and regrouping terms it is possible to show that the factors inside the curly brackets of (A3.8) may be written in the form

$$\{\ldots\} = - \sum_j \sum_{k>j} (\alpha_j \alpha_k)^{(n-3)/2} (\alpha_j + \alpha_k) f_j f_k \sum_{lm}^{\sim jk} \alpha_l \alpha_m. \tag{A3.10}$$

This gives

$$\begin{aligned} F_1 - F = &\left\{ \sum_j f_j^2 \alpha_j^{n-3} \sum_{klm}^{\sim j} \alpha_k \alpha_l \alpha_m + \sum_j \sum_k (\alpha_j \alpha_k)^{(n-3)/2} f_j f_k (\alpha_j + \alpha_k) \sum_{lm}^{\sim jk} \alpha_l \alpha_m \right\} \\ &+ \sum_j f_j^2 \alpha_j \left(\alpha_j^{n-5} \sum_{klmi}^{\sim j} \alpha_k \alpha_l \alpha_m \alpha_i + \ldots + \prod_{k \neq j} \alpha_k \right). \end{aligned} \tag{A3.11}$$

The factors inside the curly brackets of (A3.11) may be simplified using the relation

$$\left(\sum_j \alpha_j^{(n-3)/2} f_j\right)\left(\sum_j \alpha_j^{(n-3)/2} f_j \sum_{klm}^{\sim j} \alpha_k\alpha_l\alpha_m\right)=0 \tag{A3.12}$$

which is valid because of (A3.2). (A3.11) may then be simplified to give

$$F_1-F=\left\{\sum_j f_j^2\alpha_j^{n-4}\sum_{klmi}^{\sim j}\alpha_k\alpha_l\alpha_m\alpha_i+2\sum_j\sum_k f_jf_k(\alpha_j\alpha_k)^{(n-3)/2}\sum_{lmi}^{\sim jk}\alpha_l\alpha_m\alpha_i\right\}$$
$$-\sum_j f_j^2\alpha_j\left(\alpha_j^{n-6}\sum_{klmij_1}^{\sim j}\alpha_k\alpha_l\alpha_m\alpha_i\alpha_{j_1}-\ldots-\prod_{k\neq j}\alpha_k\right). \tag{A3.13}$$

Comparison of (A3.8), (A3.11) and (A3.13) shows that after each reduction using a formula like (A3.9) or (A3.12) the number of terms inside the round brackets of (A3.8) or (A3.11) is reduced by one. The next reduction may be accomplished using the relation

$$\left(\sum_j \alpha_j^{(n-5)/2} f_j\right)\left(\sum_j \alpha_j^{(n-3)/2} f_j \sum_{klmi}^{\sim j} \alpha_k\alpha_l\alpha_m\alpha_i\right)=0. \tag{A3.14}$$

It is possible to proceed as indicated until the difference between F_1 and F reduces to just two terms:

$$F_1-F=\sum_j f_j^2\alpha_j\prod_{k\neq j}\alpha_k+2\sum_j\sum_{k>j}f_jf_k\alpha_j\alpha_k\prod_{l\neq j,k}\alpha_l \tag{A3.15}$$

which can be reduced using (A3.2) to

$$F_1-F=\left(\prod_k\alpha_k\right)\left(\sum_j f_j\right)^2=0. \tag{A3.16}$$

Therefore it is possible to conclude that

$$\sum_j \alpha_j^{(n-1)/2} f_j\left(\alpha_j-\sum_{k\neq j}\alpha_k\right)=\sum_j f_j^2\alpha_j\prod_{k\neq j}(\alpha_j-\alpha_k). \tag{A3.17}$$

Case (*ii*). Let n be an odd integer. Let g_j be a set of functions which satisfy

$$\sum_{j=1}^n \alpha_j^k g_j=0 \qquad k=0,1,\ldots,\tfrac{1}{2}(n-3). \tag{A3.18}$$

Let

$$G=\left(\sum_{j=1}^n \alpha_j^{(n-1)/2}g_j\right)^2=\sum_{j=1}^n\sum_{k=1}^n(\alpha_j\alpha_k)^{(n-1)/2}g_jg_k \tag{A3.19}$$

and

$$G_1=\sum_{j=1}^n g_j^2\prod_{k\neq j}(\alpha_j-\alpha_k). \tag{A3.20}$$

Considering the difference between G and G_1 and using relations of the form

$$\left(\sum_j \alpha_j^{(n-3)/2} g_j\right)\left(\sum_j \alpha_j^{(n-1)/2} g_j \sum_{k\neq j} \alpha_k\right)=0 \tag{A3.21}$$

$$\left(\sum_j \alpha_j^{(n-3)/2} g_j\right)\left(\sum_j \alpha_j^{(n-3)/2} g_j \sum_{kl}^{\sim j} \alpha_k\alpha_l\right)=0 \tag{A3.22}$$

$$\left(\sum_j \alpha_j^{(n-5)/2} g_j\right)\left(\sum_j \alpha_j^{(n-3)/2} g_j \sum_{klm}^{\sim j} \alpha_k\alpha_l\alpha_m\right)=0 \tag{A3.23}$$

etc, which are valid because of (A3.2) it is possible to proceed as in case (i) to show that

$$G_1 - G = 0. \tag{A3.24}$$

Hence

$$\left(\sum_j \alpha_j^{(n-1)/2} g_j\right)^2 = \sum_j g_j^2 \prod_{k\neq j} (\alpha_j - \alpha_k). \tag{A3.25}$$

Case (*iii*). Let n be an even integer and g_j a set of functions which satisfy

$$\sum_{j=1}^{n} \alpha_j^k g_j = 0 \qquad k = 0, 1, \ldots, \tfrac{1}{2}(n-4) \tag{A3.26}$$

$$\sum_{j=1}^{n} \alpha_j^{(n-2)/2} g_j = 1. \tag{A3.27}$$

Let

$$\tilde{F} = \sum_{j=1}^{n} \alpha_j^{(n-2)/2} g_j \left(\alpha_j - \sum_{k\neq j} \alpha_k\right) \tag{A3.28}$$

and

$$\tilde{F}_1 = \sum_{j=1}^{n} g_j^2 \prod_{k\neq j} (\alpha_j - \alpha_k). \tag{A3.29}$$

Comparison of F and F_1 and successive elimination of the terms in the difference by the same procedure as in cases (i) and (ii) then shows that

$$\tilde{F}_1 - \tilde{F} = 0. \tag{A3.30}$$

Hence

$$\sum_j \alpha_j^{(n-2)/2} g_j \left(\alpha_j - \sum_{k\neq j} \alpha_k\right) = \sum_j g_j^2 \prod_{k\neq j} (\alpha_j - \alpha_k). \tag{A3.31}$$

Case (*iv*). Let n be an even integer and f_j a set of functions which satisfy

$$\sum_{j=1}^{n} \alpha_j^k f_j = 0 \qquad k = 0, 1, \ldots, \tfrac{1}{2}(n-2). \tag{A3.32}$$

Let

$$\tilde{G} = \left(\sum_{j=1}^{n} \alpha_j^{n/2} f_j\right)^2 = \sum_j \sum_k (\alpha_j \alpha_k)^{n/2} f_j f_k \tag{A3.33}$$

and

$$\tilde{G}_1 = \sum_{j=1}^{n} \alpha_j f_j^2 \prod_{k\neq j} (\alpha_j - \alpha_k). \tag{A3.34}$$

Proceeding as in earlier cases it is possible to show that

$$\tilde{G}_1 - \tilde{G} = 0. \tag{A3.35}$$

Hence

$$\left(\sum_j \alpha_j^{n/2} f_j\right)^2 = \sum_j f_j^2 \alpha_j \prod_{k\neq j} (\alpha_j - \alpha_k). \tag{A3.36}$$

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